

# Schwarzian derivative and Numata Finsler structures

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## Abstract

The flag curvature of the Numata Finsler structures is shown to admit a non-trivial prolongation to the one-dimensional case, revealing an unexpected link with the Schwarzian derivative of the diffeomorphisms associated with these Finsler structures.

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## 1 Finsler structures in a nutshell

### 1.1 Finsler metrics

A Finsler structure is a pair  $(M, F)$  where  $M$  is a smooth,  $n$ -dimensional, manifold and  $F : TM \rightarrow \mathbb{R}^+$  a given function whose restriction to the slit tangent bundle  $TM \setminus M = \{(x, y) \in TM \mid y \in T_x M \setminus \{0\}\}$  is strictly positive, smooth, and positively homogeneous of degree one, i.e.,  $F(x, \lambda y) = \lambda F(x, y)$  for all  $\lambda > 0$ ; one furthermore demands that the  $n \times n$  vertical Hessian matrix with entries  $g_{ij}(x, y) = (\frac{1}{2}F^2)_{y^i y^j}$  be positive definite,  $(g_{ij}) > 0$ . See [1]. These quantities are (positively) homogeneous of degree zero, and the fundamental tensor

$$g = g_{ij}(x, y) dx^i \otimes dx^j \quad (1.1)$$

defines a *sphere's worth of Riemannian metrics* on each  $T_x M$  parametrized by the direction of  $y$ . See [2].

The distinguished “vector field”

$$\ell = \ell^i \frac{\partial}{\partial x^i}, \quad \text{where} \quad \ell^i(x, y) = \frac{y^i}{F(x, y)}, \quad (1.2)$$

actually a section of  $\pi^*(TM)$  where  $\pi : TM \setminus M \rightarrow M$  is the natural projection, is such that  $g(\ell, \ell) = 1$ .

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There is a wealth of Finsler structures, apart from the special case of Riemannian structures  $(M, g)$  for which  $F(x, y) = \sqrt{g_{ij}(x)y^i y^j}$ . For instance, the so-called Randers metrics

$$F(x, y) = \sqrt{a_{ij}(x)y^i y^j} + b_i(x)y^i \quad (1.3)$$

satisfy all previous requirements if  $a = a_{ij}(x)dx^i \otimes dx^j$  is a Riemann metric and if the 1-form  $b = b_i(x)dx^i$  is such that  $a^{ij}(x)b_i(x)b_j(x) < 1$  for all  $x \in M$ .

## 1.2 Flag curvature

Unlike the Riemannian case, there is no canonical linear Finsler connection on  $\pi^*(TM)$ . An example, though, is provided by the Chern connection  $\omega_j^i = \Gamma_{jk}^i(x, y)dx^k$  which is uniquely defined by the following requirements [1]: (i) it is symmetric,  $\Gamma_{jk}^i = \Gamma_{kj}^i$ , and (ii) it *almost* transports the metric tensor, i.e.,  $dg_{ij} - \omega_i^k g_{jk} - \omega_j^k g_{ik} = 2C_{ijk}\delta y^k$ , with  $\delta y^i = dy^i + N_j^i dx^j$ , where the  $N_j^i(x, y) = \Gamma_{jk}^i y^k$  are the components of the non linear connection associated with the Chern connection, and the  $C_{ijk}(x, y) = \frac{1}{2} (g_{ij})_{y^k}$  those of the Cartan tensor, specific to Finsler geometry.

Using the “horizontal covariant derivatives”  $\delta/\delta x^i = \partial/\partial x^i - N_i^j \partial/\partial y^j$ , one expresses the (horizontal-horizontal part of the) Chern curvature by

$$R_j^i{}_{kl} = \frac{\delta}{\delta x^k} \Gamma_{jl}^i + \Gamma_{mk}^i \Gamma_{jl}^m - (k \leftrightarrow l), \quad (1.4)$$

and the *flag curvature* (associated with the flag  $\ell \wedge v$  defined by  $v \in T_x M$ ) by

$$K(x, y, v) = \frac{R_{ik} v^i v^k}{g(v, v) - g(\ell, v)^2}, \quad \text{where} \quad R_{ik} = \ell^j R_{jik\ell} \ell^\ell. \quad (1.5)$$

One says that a Finsler structure is of *scalar curvature* if  $K(x, y, v)$  does not depend on the vector  $v$ , i.e., if

$$R_{ik} = K(x, y)h_{ik}, \quad (1.6)$$

with  $h_{ik} = g_{ik} - \ell_i \ell_k$  the components of the “angular metric”, where  $\ell_i = g_{ij} \ell^j (= F_{y^i})$ . See [1, 2].

## 2 Numata Finsler structures

### 2.1 The Numata metric

Numata [4] has proved that metrics of the form  $F(x, y) = \sqrt{q_{ij}(y)y^i y^j} + b_i(x)y^i$ , on  $TM$  where  $M \subset \mathbb{R}^n$ , with  $(q_{ij}) > 0$  and  $db = 0$  are, indeed, of scalar curvature. See [2].

Of some interest is the special case  $q_{ij} = \delta_{ij}$  and  $b = df$  with  $f \in C^\infty(M)$ , viz.,

$$F(x, y) = \sqrt{\delta_{ij} y^i y^j} + f_{x^i} y^i, \quad (2.1)$$

where

$$M = \left\{ x \in \mathbb{R}^n \mid \sum_{i=1}^n f_{x^i}^2 < 1 \right\}. \quad (2.2)$$

The computation of the flag curvature of this particular Randers metric (1.3) can be found in [1] and yields

$$K(x, y) = \frac{3}{4} \frac{1}{F^4} (f_{x^i x^j} y^i y^j)^2 - \frac{1}{2} \frac{1}{F^3} f_{x^i x^j x^k} y^i y^j y^k. \quad (2.3)$$

## 2.2 Flag curvature & Schwarzian derivative

The expression (2.3) of the flag curvature of the Numata metric (2.1) holds for  $n \geq 2$ .

If  $n = 1$ , the left-hand side of (1.6) vanishes along with the curvature (1.4), while its right-hand vanishes as well since the angular metric has rank zero. For this particular dimension, Equation (1.6) trivially holds true, but tells, however, nothing about the flag curvature  $K(x, y)$ .

At this stage, it is worth noting that (2.3) indeed admits a prolongation to the one-dimensional case; it is therefore tempting to specialize its expression for  $n = 1$ .

Suppose, thus, that  $M \subset S^1$  is a nonempty open subset (2.2), so that we have  $TM \setminus M = T_+ M \sqcup T_- M$ , where  $T_\pm M = M \times \mathbb{R}_*^\pm$ . The metric (2.1) then reads

$$F(x, y) = |y| + f'(x)y, \quad (2.4)$$

using an affine coordinate,  $x$ , on  $S^1$ , with  $-1 < f'(x) < +1$  (see (2.2)); its restrictions to  $T_\pm M$  are given by  $F_\pm(x, y) = \varphi'_\pm(x)y > 0$ , where

$$\varphi'_\pm(x) = f'(x) \pm 1, \quad (2.5)$$

implying  $\varphi_\pm \in \text{Diff}_\pm(S^1)$ , with  $|\varphi'_\pm(x)| < 2$  (all  $x \in M$ ).

The Numata metric (2.4) on  $T_+ M$ , say, is thus associated, via (2.5), to orientation-preserving diffeomorphisms  $\varphi$  of  $S^1$  such that  $0 < \varphi'(x) < 2$  (all  $x \in M$ ). Given such a  $\varphi \in \text{Diff}_+(S^1)$ , the fundamental tensor (1.1) retains the form  $g = \varphi'(x)^2 dx^2$  and is, naturally, a Riemannian metric on  $M$ .

Rewriting Equation (2.3) for  $T_+ M$ , and bearing in mind that  $y = F(x, y)/\varphi'(x)$ , we readily find that  $K(x, y)$  is actually independent of  $y$ , namely

$$K(x) = -\frac{1}{2} \frac{1}{\varphi'(x)^2} S(\varphi)(x), \quad (2.6)$$

where

$$S(\varphi)(x) = \frac{\varphi'''(x)}{\varphi'(x)} - \frac{3}{2} \left( \frac{\varphi''(x)}{\varphi'(x)} \right)^2 \quad (2.7)$$

denotes the *Schwarzian derivative* [5] of the diffeomorphism  $\varphi$  of  $S^1$ . The argument clearly still holds, *mutatis mutandis*, for orientation-reversing diffeomorphisms of  $S^1$ .

We have thus proved the

**Theorem 2.1.** *The Numata Finsler structure  $(M, F)$ , with metric  $F$  given by (2.4) where  $M \subset S^1$  is defined by (2.2), induces a Riemannian metric,  $\mathbf{g}(\varphi) = \varphi^*(dx^2)$ , where  $\varphi \in \text{Diff}(S^1)$  is as in (2.5). The flag curvature (2.3) admits a prolongation to this one-dimensional case and retains the form*

$$K = -\frac{1}{2} \frac{\mathbf{S}(\varphi)}{\mathbf{g}(\varphi)}, \quad (2.8)$$

where  $\mathbf{S}(\varphi) = S(\varphi)(x)dx^2$  is the Schwarzian quadratic differential of  $\varphi \in \text{Diff}(S^1)$ .

As an illustration, the one-dimensional Numata Finsler structures of constant flag curvature are associated, through (2.5), to the solutions  $\varphi$  of (2.8) for  $K \in \mathbb{R}$ , viz.,  $\varphi_{\pm}(x) = K^{-\frac{1}{2}} \arctan(K^{\frac{1}{2}}(ax + b)/(cx + d))$  where  $a, b, c, d \in \mathbb{R}$  with  $ad - bc = \pm 1$ .

Let us mention another instance where the Schwarzian derivative is associated with curvature, namely the geometry of curves in Lorentzian surfaces of constant curvature [3].

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